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Smooth points of $\mathcal{G}or(T)$

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Dedicated to the memory of Professor Hideyuki Matsumura

Abstract

In this paper we investigate some relationships between codimension 2 Cohen–Macaulay graded rings and codimension three Artinian Gorenstein graded rings and use these to make some comments about the classifying spaces of such Gorenstein algebras having a fixed Hilbert function. © 1997 Elsevier Science B.V.

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0. Introduction

Iarrobino and Kanev [15] have introduced interesting parameter spaces for artinian graded Gorenstein algebras, and have raised several fascinating questions about these objects. A fundamental set of problems concerns the dimension and smoothness of these parameter spaces.

This paper deals with these particular problems in the case where the Gorenstein algebras in question are graded quotients of $k[x_0, x_1, x_2]$, i.e. the artinian Gorenstein algebras of embedding codimension ≤ 3. In this case there are many works on this much studied class of rings – results which grow out of the Buchsbaum–Eisenbud

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structure theorem for such algebras [4]. In fact, our work is a natural outgrowth of an earlier work in this area by Diesel [9].

We begin by briefly reviewing the fundamentals of the Iarrobino–Kanev classifying spaces and set up the notation we shall need and define the problem we study.

We then pass to the study of the Hilbert function of artinian quotients of $k[x_0, x_1]$ and introduce the notion of the *alignment character* of such a Hilbert function. This character is our basic tool for describing the dimensions of the parameter spaces mentioned above. We also review and recast the "numerical" study of codimension 2 arithmetically Cohen–Macaulay (henceforth ACM) algebras, in terms of the alignment character.

We then relate this "codimension 2 ACM" discussion to the "codimension 3 Gorenstein" case and review (and recast, in terms of the alignment character) the analogous numerical facts for these Gorenstein rings. The key ingredients here are the characterization by Stanley [22] of the Hilbert functions of such Gorenstein algebras and a construction method of Harima [14] for a special class of such algebras. A theorem of Kustin–Ulrich [16] on the resolution of the square of an ideal defining such an algebra is also reviewed.

We then state our conjectures about the dimensions of the Iarrobino–Kanev classifying spaces, verifying them for several important special cases. These extend analogous results of Iarrobino–Kanev in the codimension 3 case.

1. The classifying spaces $\mathbf{Gor}(T)$ and $\mathcal{G}or(T)$

Since the classifying spaces $\mathbf{Gor}(T)$ and $\mathcal{G}or(T)$ are not so well known, we shall take some time to recall a few basic ideas. For more details (including proofs of unsupported statements) one can consult either the paper of Iarrobino and Kanev [15] or the expository article [10].

Let $R = k[x_0, \ldots, x_n]$ (k an algebraically closed field of characteristic 0) and let $I \subset R$ be a homogeneous ideal for which $A = R/I = \bigoplus A_i$ is a graded Gorenstein artinian k-algebra with socle degree j (i.e. for which $\dim_k A_j = 1$ and $\dim_k A_{j+t} = 0$ for all $t > 0$).

Recall that Macaulay exhibited a bijective correspondence between such graded Gorenstein artinian quotients of R having socle degree j and the projective space $\mathbb{P}(S_j)$ where $S = k[y_0, \ldots, y_n]$.

The correspondence can be effected by the "apolar" duality of Macaulay, i.e. we consider the elements of the ring R as partial differential operators which operate on the elements of the ring S. This action turns S into a (non-finitely generated) R-module. Having said this, the Macaulay correspondence can be described as follows:

> if $F \in S_j$ and $I = \mathrm{ann}_R(F)$ then $A = R/I$ is the corresponding (graded) Gorenstein artinian algebra with socle degree j.

Once we observe that $\mathrm{ann}_R(F) = \mathrm{ann}_R(\lambda F)$ for any $\lambda \in k^*$, the correspondence with $\mathbb{P}(S_j)$ is then clear.

We let $\mathbf{H}(A, t) := \dim_k A_t = h_t$ denote the Hilbert function of the Gorenstein ring A above. If we continue to let j denote the socle degree of A then it is well known that

(A_1) $h_0 = h_j = 1$;

(B_1) $h_i = h_{j-i}$ for all $i = 0, \ldots, j$;

(C_1) $h_i \le \min\{\binom{i+n}{n}, \binom{j-i+n}{n}\}$ for all $i = 0, \ldots, j$.

The vector

$$\mathbf{h}(A) = (h_0, h_1, \ldots, h_{j-1}, h_j)$$

is then referred to as the **h**-*vector* of the ring A.

Now for fixed n and j one can consider arbitrary sequences of positive integers $T = (h_0, h_1, h_2, \ldots, h_{j-1}, h_j)$ which satisfy (A_1), (B_1), and (C_1) above. We shall call such a sequence a *symmetric sequence of length j*.

Apart from a theorem of Serre for the case $n = 1$ (which says that a Gorenstein quotient of $k[x_0, x_1]$ is a complete intersection) and a theorem of Stanley for the case $n = 2$ (which we shall describe below – Theorem 2.11) little is known about which symmetric sequences of length j can be the **h**-vector of a Gorenstein graded k-algebra (but see [2, 3]).

Now let $T_1 = (a_0, a_1, \ldots, a_{j-1}, a_j)$ and $T_2 = (b_0, b_1, \ldots, b_{j-1}, b_j)$ be two symmetric sequences of length j. We say that $T_1 \le T_2$ if $a_i \le b_i$ for all $i = 0, 1, \ldots, j$, and that $T_1 < T_2$ if $T_1 \le T_2$ and $a_i < b_i$ for at least one index i.

Notice that for a fixed n and j there are only a finite number of possible symmetric sequences of length j and that there is a unique minimal such sequence, namely

$$T^{(n)}_{\min} = \underbrace{(1, 1, \ldots, 1)}_{j+1\text{-times}}$$

and a unique maximal such sequence, namely

$$T^{(n)}_{\max} = \left(1, n+1, \binom{n+2}{2}, \binom{n+3}{3}, \ldots, \binom{n+3}{3}, \binom{n+2}{2}, n+1, 1\right).$$

Definition 1.1. Let T be a symmetric sequence of length j. We denote by

(i) $\quad \mathbf{Gor}(\le T) = \{[F] \in \mathbb{P}(S_j) \mid \text{if } I = \mathrm{ann}_R(F) \text{ and } A = R/I \text{ then } \mathbf{h}(A) \le T\}$

and

(ii) $\quad \mathbf{Gor}(T) = \mathbf{Gor}(\le T) \Big\backslash \left(\bigcup_{T' < T} \mathbf{Gor}(T') \right).$

Now, if F is a form of degree j in S and I is its annihilator in R, it is known (via Macaulay's inverse systems, see e.g. [10, 15] for details) that the Hilbert function of the Gorenstein ring R/I is the same as that of the R-submodule of S generated by F. We can figure out the dimensions of the graded pieces of this submodule by calculating the ranks of various catalecticant matrices. The following simple example will suffice to give the ideas.

Example 1.2. Let $F = y_0^4 + (y_0y_1)^2 + y_2^4$ in $k[y_0, y_1, y_2]$. The R-submodule of S generated by F is:

$$(F) = \langle R_4F\rangle \oplus \langle R_3F\rangle \oplus \langle R_2F\rangle \oplus \langle R_1F\rangle \oplus \langle F\rangle.$$

By symmetry, it will be enough to calculate $\dim_k R_1F$ and $\dim_k R_2F$.
Now

$$x_0F = \partial/\partial y_0(F) = 4y_0^3 + 2y_0y_1^2,$$

$$x_1F = \partial/\partial y_1(F) = 2y_0^2y_1,$$

$$x_2F = \partial/\partial y_2(F) = 4y_2^3.$$

So, $\dim_k R_1F$ is the rank of the 3×10 matrix below (whose rows are indexed by x_0, x_1, x_2 – the monomials of degree 1 in R – and whose columns are indexed by the monomials of degree 3 in S – ordered lexicographically);

$$\begin{pmatrix} 4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

This matrix has rank 3.

As for the dimension of R_2F, we first calculate:

$$x_0^2F = \frac{\partial^2}{\partial y_0 \partial y_0}(F) = 12y_0^2 + 2y_1^2,$$

$$x_0x_1F = \frac{\partial^2}{\partial y_0 \partial y_1}(F) = 4y_0y_1,$$

$$x_0x_2F = \frac{\partial^2}{\partial y_0 \partial y_2}(F) = 0,$$

$$x_1^2F = \frac{\partial^2}{\partial y_1 \partial y_1}(F) = 2y_0^2,$$

$$x_1x_2F = \frac{\partial^2}{\partial y_1 \partial y_2}(F) = 0,$$

$$x_2^2F = \frac{\partial^2}{\partial y_2 \partial y_2}(F) = 12y_2^2,$$

and then find the rank of the 6×6 matrix

$$\begin{pmatrix} 12 & 0 & 0 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 \end{pmatrix}$$

(whose rows are indexed by the monomials of degree 2 in R and whose columns are indexed by the monomials of degree 2 in S-ordered lexicographically).

Since this last matrix has rank 4 we obtain: if $I = \mathrm{ann}_R(F)$ and $A = R/I$, the **h**-vector of A is (1, 3, 4, 3, 1).

The matrices of this example are two of the various catalecticant matrices we can associate to F. They are usually referred to as $\mathrm{Cat}_F(1, 3)$ and $\mathrm{Cat}_F(2, 2)$ respectively.

This example is easily abstracted as follows: we can form a determinantal subscheme of $\mathbb{P}(S_j)$, called $\mathcal{G}or(\leq T)$, by imposing the appropriate rank conditions on the catalecticant matrices associated to the generic form of degree j in S. Moreover, by considering the scheme-theoretic difference, we can also define the schemes $\mathcal{G}or(T)$.

Notice that, with these definitions, we have

$$\mathbf{Gor}(\leq T) = \mathcal{G}or(\leq T)^{\mathrm{red}} \quad \text{and} \quad \mathbf{Gor}(T) = \mathcal{G}or(T)^{\mathrm{red}}.$$

Thus, both $\mathbf{Gor}(\leq T)$ and $\mathbf{Gor}(T)$ are reduced algebraic varieties. In particular, the $\mathbf{Gor}(\leq T)$ are all closed subvarieties of $\mathbb{P}(S_j)$ and, if $T \neq T^{(n)}_{\min}$, $\mathbf{Gor}(T)$ is open in $\mathbb{P}(S_j)$.

Example 1.3. (See also [10]).

(1) For fixed n and j we have

$$\mathbf{Gor}(\leq T^{(n)}_{\min}) = \mathbf{Gor}(T^{(n)}_{\min}) = v_j(\mathbb{P}^n) \subseteq \mathbb{P}(S_j),$$

where $v_j(\mathbb{P}^n)$ is the Veronese embedding of $\mathbb{P}^n$ in $\mathbb{P}(S_j)$. (This is so because it is easy to prove that $A = R/I$ has $h(A) \leq T^{(n)}_{\min}$ if and only if $I = \mathrm{ann}_R(F)$ where $F = L^j$ (L a linear form in S_1).) Pucci has recently shown that $\mathcal{G}or(T^{(n)}_{\min}) = v_j(\mathbb{P}^n)$, i.e. that the scheme $\mathcal{G}or(T^{(n)}_{\min})$ is a reduced scheme! This will be the subject of another work.

(2) Fix n and j and let $T^{(n)}_2 = (1, 2, 2, \ldots, 2, 2, 1)$. One can show that in this case $\mathbf{Gor}(\leq T^{(n)}_2)$ is the secant (line) variety to $v_j(\mathbb{P}^n)$ in $\mathbb{P}(S_j)$, denoted $Sec_1(v_j(\mathbb{P}^n))$. Notice that this is a singular variety in $\mathbb{P}(S_j)$ whose singular points are precisely $\mathbf{Gor}(T^{(n)}_{\min})$. I.e. in this case the points of $\mathbf{Gor}(T^{(n)}_2)$ are the smooth points of $\mathbf{Gor}(\leq T^{(n)}_2)$.

(3) A more elaborate example is provided by considering all symmetric sequences of length 4 which can be the **h**-vector of a Gorenstein quotient of $k[x_0, x_1, x_2]$ (see Theorem 2.11).

In this case the sequences are linearly ordered by

$$T_1 = (1, 3, 6, 3, 1), \quad T_2 = (1, 3, 5, 3, 1), \quad T_3 = (1, 3, 4, 3, 1),$$

$$T_4 = (1, 3, 3, 3, 1), \quad T_5 = (1, 2, 2, 2, 1), \quad T_6 = (1, 1, 1, 1, 1).$$

We have already commented on $\mathbf{Gor}(T_6)$ and $\mathbf{Gor}(\leq T_5)$.

Now (see [10]) for details we have:

$\mathbf{Gor}(\leq T_4) \supseteq Sec_2(v_4(\mathbb{P}^2))$

(the variety of secant planes to $(v_4(\mathbb{P}^2))$, a variety of dimension 8);

$\mathbf{Gor}(\leq T_3) \supseteq Sec_3(v_4(\mathbb{P}^2))$

(the variety of secant $\mathbb{P}^3$'s to $(v_4(\mathbb{P}^2))$, a variety of dimension 11):

$\mathbf{Gor}(\leq T_2) \supseteq Sec_4(v_4(\mathbb{P}^2))$

(the variety of secant $\mathbb{P}^4$'s to $(v_4(\mathbb{P}^2))$, a variety of dimension 13, one of the deficient secant varieties to $v_4(\mathbb{P}^2)$).

In order to describe the schemes $\mathcal{G}or(\leq T_i)$ (for the T_i above) let

$$F = ay_0^4 + by_0^3y_1 + cy_0^3y_2 + dy_0^2y_1^2 + ey_0^2y_1y_2 + fy_0^2y_2^2 + gy_0y_1^3 + hy_0y_1^2y_2 + ky_0y_1y_2^2 + ly_0y_2^3 + my_1^4 + py_1^3y_2 + ry_1y_2^3 + sy_2^4$$

be a general form of degree 4 in S_4 ($S = k[y_0, y_1, y_2]$).

The two relevant catalecticant matrices are:

$$A = \begin{pmatrix} a & b & c & d & e & f & g & h & k & l \\ b & d & e & g & h & k & m & p & q & r \\ c & e & f & h & k & l & p & q & r & s \end{pmatrix} = Cat_F(1,3),$$

and

$$B = \begin{pmatrix} a & b & c & d & e & f \\ b & d & e & g & h & k \\ c & e & f & h & k & l \\ d & g & h & m & p & q \\ e & h & k & p & q & r \\ f & k & l & q & r & s \end{pmatrix} = Cat_F(2,2).$$

Note. For simplicity in the exposition (and since we are assuming that the characteristic of k is 0) in writing these catalecticants we have suppressed the coefficients that arise from differentiation. That this creates no problem, in characteristic 0, is explained in [10, Section 9].

If we let $I_t(M)$ be the ideal generated by the $t \times t$ minors of the matrix M, then:

$\mathcal{G}or(\leq T_2)$ is defined by $I_6(B)$, $\mathcal{G}or(\leq T_3)$ is defined by $I_5(B)$,

$\mathcal{G}or(\leq T_4)$ is defined by $I_4(B)$, $\mathcal{G}or(\leq T_5)$ is defined by $I_3(B) + I_3(A)$,

$\mathcal{G}or(T_6)$ is defined by $I_2(B) + I_2(A)$.

In view of Pucci's recent work, cited above, $I_2(B) = I_2(A)$ is the defining ideal of $v_4(\mathbb{P}^2)$. So, $\mathcal{G}or(T_6) = \mathbf{Gor}(T_6) = v_4(\mathbb{P}^2)$.

Since $I_6(B) = \det B$ (known to be the irreducible polynomial of degree 6 which defines $Sec_4(v_4(\mathbb{P}^2))$), we also have

$$\mathbf{Gor}(\leq T_2) = \mathcal{G}or(\leq T_2) = Sec_4(v_4(\mathbb{P}^2)).$$

There are many interesting questions one can ask about these classifying spaces and the first significant steps towards understanding them have been taken by Iarrobino and Kanev in [15] and Diesel in [9]. It is a well-known problem to try and determine even when $\mathbf{Gor}(T) \neq \emptyset$!

Iarrobino and Kanev [15, Theorem 2.5] have proved a wonderful theorem which gives the dimension of the tangent space to the scheme $\mathcal{G}or(T)$ at any of its closed points. More precisely (and with the notation of this section):

Theorem 1.4 (Iarrobino and Kanev [15]). *Let $F \in S_j$ and let $I = \mathrm{ann}_R(F) \subseteq R$. Let $T = \mathbf{h}(A)$ be the* **h***-vector of the Gorenstein ring $A = R/I$ and let $\mathcal{T}_F$ be the tangent space to the scheme $\mathcal{G}or(T)$ at the point $[F]$. Then*

$$\dim_k \mathcal{T}_F + 1 = \dim_k (R/I^2)_j.$$

Thus, in view of this theorem, it is important to be able to calculate the dimension, as vector space, of the degree j piece of the ideal I^2.

We propose to do this for some $I \subseteq k[x_0, x_1, x_2]$, i.e. for some codimension 3 Gorenstein ideals. Our methods involve making a detailed study of certain codimension 2 Cohen–Macaulay ideals and using these results (with liaison) to say something about the codimension 3 Gorenstein case.

2. Hilbert functions of codim 2 Cohen–Macaulay and codim 3 Gorenstein rings

It is well-known that the study of Hilbert functions of arithmetically Cohen–Macaulay quotients, by height two ideals of $k[x_0, x_1, \ldots, x_n]$, is equivalent to the study of Hilbert functions of quotients of $k[x_0, x_1]$ by ideals I having radical (x_0, x_1).

If $A = k[x_0, x_1]/I = \oplus A_i$ is such a k-algebra and if we let $\mathbf{H}(A, t) = \dim_k A_t = a_t$, then the (infinite) sequence of integers

$$a_0, a_1, \ldots, 0, 0, \ldots$$

is the *Hilbert function* of the ring A. (Since $\sqrt{(I)} = (x_0, x_1)$, the Hilbert function is eventually 0.)

We denote by $\sigma = \sigma(I)$ (or $\sigma(A)$) the least integer for which

$$\mathbf{H}(A, \sigma) = 0 \quad \text{and} \quad \mathbf{H}(A, \sigma - 1) \neq 0.$$

It follows that $\mathbf{H}(A, t) = 0$ for all $t \geq \sigma$. The least integer t for which $I_t \neq 0$ is denoted $\alpha(I)$ (or $\alpha(A)$).

With this notation, the vector of positive integers

$$T = (a_0, a_1, \dots, a_{\sigma(I)-1})$$

contains all information on the Hilbert function of A. This is the **h**-*vector* of the ring A.

It is well-known (see e.g. [7]) that the entries of the **h**-vector of A (as above) satisfy:

(A) $a_i \leq i + 1$ for all i, with equality $\Leftrightarrow$ $0 \leq i \leq \alpha(I) - 1$;

(B) $a_i \geq a_{i+1}$ for $\alpha(I) - 1 \leq i$.

Furthermore, any vector of positive integers which satisfies (A) and (B) is the **h**-vector of some artinian quotient of $k[x_0, x_1]$.

This permits us to discuss the vectors of this type as Hilbert functions without ever specifying any ring! Consequently, if $T = (1, a_1, a_2, \dots, a_n)$ is a vector of positive integers satisfying (A) and (B) we can speak of $\alpha(T)$ and $\sigma(T)$. Moreover, given such a T, the vector T' made from the positive integers in the sequence

$$a_1 - 1,\ a_2 - 1, \dots, a_n - 1$$

also satisfies (A) and (B), and thus we can write

$$T' = (1, b_1, \dots, b_{n'}) \quad \text{with } n' \leq n - 1.$$

If $n' \geq 1$ we can repeat the process. In fact, the process can be repeated exactly $\alpha(T) - 1$ times and the resulting sequence of integers – $\sigma(T), \sigma(T'), \dots$ – is a strictly decreasing family of integers. If we reorder these integers in **increasing** order we obtain, what we shall call, the *alignment character* of T.

Example 2.1. Let $T = (1, 2, 3, 4, 3, 3, 1)$. Then $\alpha(T) = 4$ and $\sigma(T) = 7$. Now $T' = (1, 2, 3, 2, 2)$ and $\sigma(T') = 5$. Repeating we get $T'' = (1, 2, 1, 1)$ and $\sigma(T'') = 4$. Repeating the procedure once again we get $T''' = (1)$ and $\sigma(T''') = 1$. Thus, the alignment character of T is (1, 4, 5, 7).

If we are given an alignment character $(d_1, \dots, d_m)$, the following procedure shows how to construct a vector $T = (1, t_1, \dots, t_{d_m - 1})$ satisfying (A) and (B) and having alignment character $(d_1, \dots, d_m)$.

Let σ^d denote the infinite sequence consisting of d 1's followed by 0's. Write down the sequences $\sigma^{d_1}, \dots, \sigma^{d_m}$ (successively shifting them to the left) and add, i.e.

$$\begin{array}{llllllllllll}
\sigma^{d_1} & : & & & 1 & 1 & \cdots & 1 & 0 & \to & & \\
\sigma^{d_2} & : & & 1 & 1 & & \cdots & & 1 & 0 & \to & \\
\vdots & & & & & & & & & & & \\
\sigma^{d_m} & : & 1 & 1 & & & \cdots & & & 1 & 0 & \to \\
\hline
T & : & 1 & 2 & t_2 & & \cdots & & & & &
\end{array}$$

One verifies that

$$t_i = \sum_{j=1}^{m} \sigma^{d_j}(j + i - m).$$

Thus, there is a 1-1 correspondence between vectors of positive integers satisfying (A) and (B) and alignment characters.

Now let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s (distinct) points in $\mathbb{P}^2$ where $P_i \leftrightarrow \wp_i = (L_{i1}, L_{i2}) \subseteq k[x_0, x_1, x_2] = R$ (the L_{ij} linear forms). Then

$$I = I(\mathbb{X}) = \wp_1 \cap \cdots \cap \wp_s$$

and $A = R/I = \bigoplus A_i$ is the homogeneous coordinate ring of $\mathbb{X}$.

The Hilbert function of $\mathbb{X}$ (or of $A = R/I$) is the function

$$\mathbf{H}(A, t) = \mathbf{H}_{\mathbb{X}}(t) := \dim_k A_t.$$

It is well-known (and easy to prove) that if L is a linear form which describes a line in $\mathbb{P}^2$ which misses all the points of $\mathbb{X}$ then $\bar{L}$ is not a divisor of zero in the ring A. We can, by making a linear change of variables if necessary, assume that $L = x_2$. Then if we write $B = A/\bar{L}A$, we have $B \simeq k[x_0, x_1]/J$ where $\sqrt{J} = (x_0, x_1)$ and

$$\mathbf{H}(B, t) = \mathbf{H}(A, t) - \mathbf{H}(A, t-1) := \Delta\mathbf{H}(A, t)$$

is the *first difference* of the Hilbert function of A (or $\mathbb{X}$). (Higher differences are formed in the obvious way.)

Since $B \simeq k[x_0, x_1]/J$, $\sqrt{J} = (x_0, x_1)$, we can (and will) speak of the alignment character of the **h**-vector of B. If no confusion can occur, we call it also the alignment character of $\mathbf{H}_{\mathbb{X}}$. We also write $\alpha(\mathbb{X})$ and $\sigma(\mathbb{X})$for $\alpha(B)$ and $\sigma(B)$.

We now explain where the name "alignment character" comes from.

Proposition 2.2. *Let $\mathbf{H}_{\mathbb{X}}$ be the Hilbert function of a set of s distinct points in $\mathbb{P}^2$ and let $(d_1, d_2, \dots, d_m)$ be the alignment character of $\mathbf{H}_{\mathbb{X}}$.*

There is another set, $\mathbb{Y}$, *of s points in* $\mathbb{P}^2$, *with* $\mathbf{H}_{\mathbb{X}} = \mathbf{H}_{\mathbb{Y}}$ *and such that the points of* $\mathbb{Y}$ *are distributed among m distinct lines* $\mathbb{L}_1, \mathbb{L}_2, \ldots, \mathbb{L}_m$ *with:*

d_1 *points on the line* $\mathbb{L}_1$;

d_2 *points on the line* $\mathbb{L}_2$;

$\ldots$;

d_m *points on the line* $\mathbb{L}_m$.

Proof. See [11]. □

Example 2.3. It is very easy to show that 8 points, $\mathbb{X}$, on an irreducible conic in $\mathbb{P}^2$ have Hilbert function:

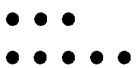
$$\mathbf{H}_{\mathbb{X}}: 1 \ 3 \ 5 \ 7 \ 8 \ 8 \ \cdots$$

i.e.

$$\Delta\mathbf{H}_{\mathbb{X}}: 1 \ 2 \ 2 \ 2 \ 1 \ 0 \ \cdots .$$

A line of $\mathbb{P}^2$ contains none, one or two points of $\mathbb{X}$.

Nevertheless, the alignment character of $\mathbf{H}_{\mathbb{X}}$ is (3, 5) and the following 8 points, $\mathbb{Y}$,



have $\mathbf{H}_{\mathbb{Y}} = \mathbf{H}_{\mathbb{X}}$.

With this proposition in mind, Roberts and Roitman [20] introduced the following definition:

Definition 2.4. A k-configuration is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ which satisfies the following conditions:

there exist integers $1 \le d_1 < \cdots < d_m$, and subsets $\mathbb{X}_1, \ldots, \mathbb{X}_m$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1, \ldots, \mathbb{L}_m \subseteq \mathbb{P}^2$ such that:

(1) $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{X}_i$;

(2) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subset \mathbb{L}_i$ for each $i = 1, \ldots, m$, and;

(3) $\mathbb{L}_i$ $(1 < i \le m)$ does not contain any points of $\mathbb{X}_j$ for all $j < i$.

In this case, the k-configuration in $\mathbb{P}^2$ is said to be of type $(d_1, \ldots, d_m)$.

It is easy to show that if $\mathbb{X}$ is a k-configuration of type $(d_1, \ldots, d_m)$ and if we construct the vector T satisfying (A) and (B) above and which corresponds to the alignment character $(d_1, \ldots, d_m)$, then

$$T = (\Delta\mathbf{H}(\mathbb{X}, 0), \Delta\mathbf{H}(\mathbb{X}, 1), \ldots, \ldots, \ldots).$$

Remark 2.5 (Roberts and Roitman [20]). If we let $\mathscr{S}^d$ denote the infinite sequence $1, 2, \ldots, d, d, d, \ldots$ and mimic the discussion after Example 2.1 we find that any two k-configurations of type $(d_1, \ldots, d_m)$ have the same Hilbert function, denoted $\mathbf{H}^{(d_1, \ldots, d_m)}$, where

$$\mathbf{H}^{(d_1, \ldots, d_m)}(i) = \sum_{j=1}^{m} \mathscr{S}^{d_j}(j + i - m).$$

It is possible to use the type of a k-configuration, $\mathbb{X}$, to describe the minimal free resolution of $I = I(\mathbb{X})$.

We use the notation $C.I.(a, b)$ for any set of $s = ab$ points of $\mathbb{P}^2$ which are all the points of intersection of two curves defined, respectively, by forms of degrees a and b. We write $\nu(I)$ to denote the minimal number of generators of the ideal I.

Theorem 2.6. *Let $\mathbb{X}$ be a k-configuration of type $(d_1, \ldots, d_m)$ and let $I = I(\mathbb{X})$. Then $\nu(I) = m + 1$ and the minimal free resolution of I, as an R-module, is*

$$0 \to R(-(d_1 + m)) \oplus \cdots \oplus R(-(d_i + m - i + 1)) \oplus \cdots \oplus R(-(d_m + 1))$$
$$\to R(-m) \oplus R(-(d_1 + m - 1)) \oplus \cdots \oplus R(-(d_i + m - i)) \oplus \cdots \oplus R(-d_m)$$
$$\to I \to 0.$$

Remark 2.7. By Dubriel's first theorem [7] we always have $\nu(I(\mathbb{X})) \leq \alpha(\mathbb{X}) + 1 = m + 1$. Thus, Theorem 2.6 says that the number of generators is the maximum possible. That fact alone is enough to specify the degrees of the minimal generators of $I(\mathbb{X})$ (and hence the entire resolution) Migliore has pointed out to us that a k-configuration of type $(d_1, \ldots, d_{m-1}, d_m)$ is linked (by a basic double link) to a k-configuration of type $(d_1, \ldots, d_{m-1})$ (see e.g. Migliore's book [17] or [7, Lemma 4.3]). Using that observation, one can prove (by induction) that $\nu(I(\mathbb{X})) = m + 1$ and get the conclusion of Theorem 2.6 as a consequence.

However, an elementary proof (which we now give) is also possible.

Proof of Theorem 2.6. We prove (by induction on m) that I has $m + 1$ minimal generators $F_0, \ldots, F_m$ where

$$\deg F_0 = m, \quad \deg F_1 = d_1 + m - 1, \ldots, \deg F_i = d_i + m - i, \ldots, \deg F_m = d_m.$$

Let $m = 1$. Then $\mathbb{X}$ is a k-configuration of type (d_1), i.e., $\mathbb{X}$ is a finite set of points which lie on a line $\mathbb{L}_1$. Clearly $\nu(I) = 2$ since $I = \langle F_0, F_1 \rangle$ for $F_0, F_1 \in I$ with $\deg F_0 = 1$, $\deg F_1 = d_1$, and we are done in this case.

Now assume $m > 1$. Since $\mathbb{X}$ is a k-configuration of type $(d_1, \ldots, d_m)$, there exist subsets $\mathbb{X}_1, \ldots, \mathbb{X}_m$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1, \ldots, \mathbb{L}_m$ as in Definition 2.4.

Let $\mathbb{Y} = \bigcup_{i=1}^{m-1} \mathbb{X}_i$. Then $\mathbb{Y}$ is also a k-configuration, now of type $(d_1, \ldots, d_{m-1})$. Hence, by induction, there exist $F'_0, F'_1, \ldots, F'_{m-1} \in I(\mathbb{Y})$ with degrees

$$\deg F'_0 = m-1, \quad \deg F'_1 = d_1 + (m-1) - 1, \ldots, \deg F'_{m-1} = d_{m-1}$$

such that $I(\mathbb{Y}) = \langle F'_0, F'_1, \ldots, F'_{m-1} \rangle$.

Let L_m be the linear form defining $\mathbb{L}_m$ and set $S = R/(L_m)$ and $J = (I + (L_m))/(L_m)$. Since $(L_m) \cap I = L_m \cdot (I : L_m)$, we have

$$\frac{I}{L_m \cdot [I : L_m]} = \frac{I}{(L_m) \cap I}.$$

By the 3rd isomorphism theorem,

$$\frac{I}{(L_m) \cap I} \simeq \frac{I + (L_m)}{(L_m)} = J \subset S$$

and so there is an exact sequence of graded modules

$$\begin{array}{c} 0 \to [I : L_m](-1) \xrightarrow{\times L_m} I \to \dfrac{I + (L_m)}{(L_m)} \to 0. \\ \qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad J \end{array} \tag{1}$$

Let $\mathbb{Y} = \{P_1, \ldots, P_s\}$ and $\mathbb{X}_m = \{P_{s+1}, \ldots, P_{s+t}\}$. Since

$$[\wp_i : L_m] = \begin{cases} R_j, & \text{if } L_m \in \wp_i \Leftrightarrow P_i \in \mathbb{L}_m, \\ \wp_i, & \text{if } L_m \notin \wp_i \Leftrightarrow P_i \notin \mathbb{L}_m, \end{cases}$$

for every $i = 1, \ldots, s+t$, we have

$$I : L_m = \left[\bigcap_{i=1}^{s+t} \wp_i \right] : L_m = \bigcap_{i=1}^{s+t} [\wp_i : L_m] = \bigcap_{i=1}^{s} [\wp_i : L_m] = \bigcap_{i=1}^{s} \wp_i = I(\mathbb{Y}).$$

Thus we can rewrite the exact sequence (1) as:

$$0 \to I(\mathbb{Y})(-1) \xrightarrow{\times L_m} I \to J \to 0. \tag{2}$$

It follows from (2) that

$$\mathbf{H}(S/J, t) = \begin{cases} 1 & \text{for } t = 0, \\ \mathbf{H}(R/I, t) - \dim_k(R_{t-1}/(I(\mathbb{Y}))_{t-1}) & \text{for } t \geq 1. \end{cases}$$

Applying Remark 2.5, we get

$$\mathbf{H}(S/J, t) = \mathbf{H}(\mathbb{X}_m, t) \quad \text{for all } t \geq 0.$$

Thus $I + (L_m) = I(\mathbb{X}_m)$ and, by induction, $I(\mathbb{X}_m) = \langle L_m, F_m \rangle$ (where we may as well choose $F_m \in I$).

Claim. *$I = \langle F_0, F_1, \ldots, F_m \rangle$ where $F_0 = F'_0 L_m, \ldots, F_{m-1} = F'_{m-1} L_m$.*

Proof. The inclusion $\langle F_0, F_1, \ldots, F_m \rangle \subseteq I$ is clear.

Conversely, for every $F \in I$, $\bar{F} \in J = \langle \bar{F}_m \rangle$. Hence

$$F = F_m G_m + L_m K$$

for some G_m, $K \in R$.

Since $K \in [I : L_m] = I(\mathbb{Y}) = \langle F'_0, F'_1, \ldots, F'_{m-1} \rangle$, we can write

$$K = F'_0 G_0 + F'_1 G_1 + \cdots + F'_{m-1} G_{m-1}$$

for some $G_i \in R$. Hence

$$\begin{aligned} F &= F_m G_m + L_m K \\ &= F_m G_m + L_m (F'_0 G_0 + F'_1 G_1 + \cdots + F'_{m-1} G_{m-1}) \\ &= F_m G_m + (F'_0 L_m) G_0 + (F'_1 L_m) G_1 + \cdots + (F'_{m-1} L_m) G_{m-1} \\ &= F_0 G_0 + F_1 G_1 + \cdots + F_m G_m \end{aligned}$$

which completes the proof of the claim. □

Since the Hilbert function of $\mathbb{X}$ is known from Remark 2.5 and the degrees of the minimal generators of $I(\mathbb{X})$ are known, the rest of the resolution is determined (see e.g. [6] or [12]). □

We shall need, in the course of this paper, a small generalization of the idea of a k-configuration.

Definition 2.8. A weak k-configuration is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ which satisfies the following conditions:

there exist integers $1 \le d_1 \le \cdots \le d_m$, and subsets $\mathbb{X}_1, \ldots, \mathbb{X}_m$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1, \ldots, \mathbb{L}_m \subseteq \mathbb{P}^2$ such that:

(1) $i \le d_i$ for each $i = 1, \ldots, m$;

(2) $\mathbb{X} = \bigcup_{i=1}^m \mathbb{X}_i$;

(3) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subset \mathbb{L}_i$ for each $i = 1, \ldots, m$, and;

(4) $\mathbb{L}_i$ $(1 < i \le m)$ does not contain any points of $\mathbb{X}_j$ for all $j < i$.

In this case, the weak k-configuration in $\mathbb{P}^2$ is said to be of type $(d_1, \ldots, d_m)$.

Remark 2.9. It is the case that any two weak k-configurations of type $(d_1, \ldots, d_m)$ have the same Hilbert function. See Remark 2.5 for the formula.

It is also possible, in an important special case, (although not in general) to describe the resolution of the ideal of a weak k-configuration in terms of its type.

Theorem 2.10. *Let* $\mathbb{X}$ *be a weak k-configuration of type* $(d_1, \ldots, d_m, \ldots, d_{m+l})$ *where* $d_1 < \cdots < d_m = \cdots = d_{m+l}$ *and* $l \geq 1$ *and let* $I = I(\mathbb{X})$.

If $\mathbb{X}$ *is a subset of a complete intersection in* $\mathbb{P}^2$ *of type* $(m + l, d_m)$ *then* $v(I) = m + 1$ *and the minimal free resolution of I, as an R-module, is*

$$0 \to R(-(d_1 + m + l)) \oplus \cdots \oplus R(-(d_i + m + l - i + 1)) \oplus \cdots \oplus$$
$$R(-(d_{m-1} + l + 2)) \oplus R(-(d_m + l + 1))$$
$$\to R(-(m+l)) \oplus R(-(d_1 + m + l - 1)) \oplus \cdots \oplus R(-(d_i + m + l - i)) \cdot \oplus \cdots \oplus$$
$$R(-(d_{m-1} + l + 1)) \oplus R(-d_m)$$
$$\to I \to 0$$

Proof. Let $\mathbb{L}_1, \ldots, \mathbb{L}_{m-1}$ be the first $m - 1$ lines of the weak configuration. Then, if $\mathbb{Z}$ denotes the complete intersection of the hypothesis and $\mathbb{Y} = \mathbb{Z} \backslash \mathbb{X}$, then $\mathbb{Y}$ is a k-configuration, supported on the $m - 1$ lines above, of type $(d_m - d_{m-1}, d_m - d_{m-2}, \ldots, d_m - d_1)$.

If we let J be the defining ideal of $\mathbb{Y}$ then, by Theorem 2.6, $v(J) = m$ and the degrees of the generators of J are:

$$m - 1, \quad d_m - d_{m-1} + m - 2, \ldots, d_m - d_2 + 1, \quad d_m - d_1.$$

Since the forms of degrees d_m and $m + l$ which form the complete intersection are not minimal generators of $\mathbb{Y}$ and $\mathbb{X}$ is linked to $\mathbb{Y}$ by them, it follows that $v(I) = m + 1$ [18] and the degrees of the generators of I are:

$$m + l \quad d_1 + m + l - 1 \quad d_2 + m + l - 2 \quad \cdots \quad d_i + m + l - i \quad \cdots \quad d_{m-1} + l + 1 \quad d_m.$$

This is enough to prove the theorem. □

We now turn to a discussion of the Hilbert function of an artinian Gorenstein standard graded k-algebra. We shall see, shortly, how the notion of a k-configuration enters into such a discussion.

As in Section 1, let $R = k[x_0, \ldots, x_n]$ and let I be a homogeneous ideal of R for which $A = R/I = \bigoplus A_i$ is an artinian Gorenstein ring with socle degree j and let $\mathbf{h}(A) = (h_0, h_1, \ldots, h_{j-1}, h_j)$ be the $\mathbf{h}$-vector of A. Recall also the conditions (A_1), (B_1) and (C_1) satisfied by $\mathbf{h}(A)$. In 1978 Stanley proved (using the notation above) that:

Theorem 2.11 (Stanley [22]). *If* $h_1 \leq 3$ *and* $t = \lfloor j/2 \rfloor$ *then* $(h_0, \ldots, h_j)$ *is the* $\mathbf{h}$*-vector of a Gorenstein artinian quotient of* $k[x_0, x_1, x_2]$ $\Leftrightarrow$ *the components of the vector satisfy* A_1, B_1 *and* C_1 *above (for* $n = 2$*) and the non-zero elements of the sequence* $h_0, h_1 - h_0, \ldots, h_t - h_{t-1}$ *form the* $\mathbf{h}$*-vector of some artinian quotient of* $k[x_0, x_1]$.

Thus, if we are given the **h**-vector of some Gorenstein artinian quotient of $k[x_0, x_1, x_2]$, this automatically gives us the **h**-vector of some artinian quotient of $k[x_0, x_1]$ and thus, an alignment character.

Example 2.12. Let $T = (1, 3, 5, 7, 9, 9, 9, 9, 7, 5, 3, 1)$. By Stanley's theorem, this is the **h**-vector of some artinian Gorenstein quotient of $k[x_0, x_1, x_2]$ having socle degree 11. The number $t = \lfloor 11/2 \rfloor = 5$ and so we must consider the sequence 1, 2, 2, 2, 2, 0. This sequence gives the vector $T' = (1, 2, 2, 2, 2)$. T' is the **h**-vector of an artinian quotient of $k[x_0, x_1]$ with alignment character (4, 5).

Conversely, if we are given an alignment character $(d_1, d_2, \ldots, d_m)$, this determines an **h**-vector T' of an artinian quotient of $k[x_0, x_1]$, say $T' = (a_0, a_1, \ldots, a_{d_m - 1})$. Now, if j is any integer, where $j \geq 2(d_m - 1)$, then T' and j determine, in a unique way, the **h**-vector, T, of some Gorenstein artinian quotient of $k[x_0, x_1, x_2]$ having socle degree j.

The following example illustrates this simple process.

Example 2.13. (a) Let (1, 3, 4, 5) be the alignment character and let $j = 8 = 2(5 - 1)$. This character determines the **h**-vector $T' = (1, 2, 3, 4, 3)$ of an artinian quotient of $k[x_0, x_1]$. We can "integrate" T' and use $j = 8$ to form $T = (1, 3, 6, 10, 13, 10, 6, 3, 1)$ which is the **h**-vector of some Gorenstein artinian quotient of $k[x_0, x_1, x_2]$ with socle degree 8.

(b) With the same alignment character, now choose $j = 11 > 2(5 - 1)$. We get the same T' as above, but now our choice of j gives: $T = (1, 3, 6, 10, 13, 13, 13, 13, 10, 6, 3, 1)$.

Up to this point, our discussion of codimension 3 Gorenstein Hilbert functions has been only "numeric". The remarkable thing is that these numerical manipulations with the numbers of putative **h**-vectors can be translated into constructions with rings in such a way that the putative **h**-vectors become **h**-vectors of rings!

Our discussion from now up to (and including) the next theorem recalls some interesting observations of Harima in [14].

Let $1 \leq a \leq b$ be integers. We choose two sets of distinct "parallel" lines in $\mathbb{P}^2$, $\mathbb{L}_{11}, \mathbb{L}_{12}, \ldots, \mathbb{L}_{1a}$ (the "horizontal" lines) and $\mathbb{M}_{21}, \mathbb{M}_{22}, \ldots, \mathbb{M}_{2b}$ (the "vertical" lines) which meet in exactly ab distinct points. We picture such a collection of points as a grid with a rows and b columns. Harima calls the points of this grid a *basic configuration* in $\mathbb{P}^2$ of type (a, b).

Clearly, if $1 \leq d_1 < d_2 < \cdots < d_m \leq a \leq b$, we can always find a k-configuration of type $(d_1, d_2, \ldots, d_m)$ in the lower left corner of a basic configuration of type (a, b). In that case we say the k-configuration is *embedded* in the basic configuration.

Example 2.14. Let $d_1 = 1$, $d_2 = 3$, $d_3 = 4$, $a = 5$, $b = 6$. Then the basic configuration of type (5, 6) consists of the 30 points below with the k-configuration of type (1, 3, 4) embedded in it (the "circles"):

```
*  *  *  *  *  *
*  *  *  *  *  *
●  *  *  *  *  *
●  ●  ●  *  *  *
●  ●  ●  ●  *  *
```

Notice that the complement of the embedded k-configuration is a weak k-configuration of the type we considered in Theorem 2.10.

Let $T = (h_0, h_1, \ldots, h_j)$ be a vector of positive integers which satisfies Stanley's theorem (2.11) above. We now explain how Harima uses the ideas above to construct a very specific Gorenstein ring A with **h**-vector T.

Let $t = \lfloor j/2 \rfloor$. Then, from our work above, the positive integers in h_0, $h_1 - h_0, \ldots, h_t - h_{t-1}$ determine an alignment character $(d_1, \ldots, d_m)$ where $d_m - 1 \leq t$. Form a k-configuration of points of type $(d_1, \ldots, d_m)$ (call it $\mathbb{X}$) and embed it in a basic configuration of type (d_m, b) where $b = j + 3 - d_m$. Let $\mathbb{Y}$ be the complement of $\mathbb{X}$ in this basic configuration. (In general, $\mathbb{Y}$ is not a k-configuration but rather a weak k-configuration.)

Theorem 2.15 (Harima [14]). *With the notation above; if $J = I(\mathbb{X}) + I(\mathbb{Y})$ then $A = k[x_0, x_1, x_2]/J$ is an artinian Gorenstein ring and*

$$\mathbf{H}(A, i) = \begin{cases} \mathbf{H}(\mathbb{X}, i) & \text{for } i = 0, \ldots, a + b - 2 - d_m, \\ \mathbf{H}(\mathbb{X}, a + b - 3 - i) & \text{for } i = a + b - 1 - d_m, \ldots, a + b - 3, \end{cases}$$

i.e. the **h**-*vector of the Gorenstein ring A is T.*

We illustrate this theorem with the following example.

Example 2.16. Consider $T = (1, 3, 5, 7, 9, 9, 9, 9, 7, 5, 3, 1)$ of Example 2.12. The procedure above tells us to form a k-configuration of type (4, 5) and embed it in a basic configuration of type $(5, b)$, $b = 11 + 3 - 5 = 9$.

Thus, in the diagram below, the points of the k-configuration, $\mathbb{X}$, of type (4, 5) are given by circles and those of the complement, $\mathbb{Y}$ (a weak k-configuration of type (4, 5, 9, 9, 9)) are given by $*$.

```
*  *  *  *  *  *  *  *  *
*  *  *  *  *  *  *  *  *
*  *  *  *  *  *  *  *  *
●  ●  ●  ●  *  *  *  *  *
●  ●  ●  ●  ●  *  *  *  *
```

Then $J = I(\mathbb{X}) + I(\mathbb{Y}) \subseteq k[x_0, x_1, x_2] = R$ is an ideal for which R/J is a Gorenstein artinian ring with **h**-vector T.

We add an important additional observation to Harima's construction, i.e. we give, in terms of the socle degree and the alignment character, a minimal set of generators for the ideal $I(\mathbb{X}) + I(\mathbb{Y})$ constructed by Harima.

To fix the notation, let $\mathbb{X}$ be a k-configuration of type $(d_1, \dots, d_m)$ which is embedded in a basic configuration, $\mathbb{Z}$, of type (a, b) (where $a = d_m < b$).

Let the "horizontal" lines of the basic configuration be $\mathbb{L}_1, \dots, \mathbb{L}_{d_m}$ (with defining equations $L_1, \dots, L_{d_m}$) and the "vertical" lines be $\mathbb{M}_1, \dots, \mathbb{M}_b$ (with defining equations $M_1, \dots, M_b$).

There are two separate (but very similar) cases to consider:

Case 1: $a = d_m > m$. In this case, let

$$f_1 = L_{d_m} \cdots L_{d_m - (m-1)} \quad (\deg f_1 = m)$$

$$g_1 = M_1 \cdots M_{d_m} \quad (\deg g_1 = d_m).$$

(Note that f_1 and g_1 are part of a minimal generating set for $I(\mathbb{X})$ and also a regular sequence of length 2).

Let

$$g_2 = \prod_{i=1}^{b} M_i \quad \text{and} \quad f_2 = \prod_{i=1}^{d_m} L_i$$

and let $\mathbb{Y}$ be the complement of $\mathbb{X}$ in $\mathbb{Z}$. Then $\mathbb{Y}$ is a weak k-configuration of type

$$(b - d_m, b - d_{m-1}, \dots, b - d_1, b, \dots, b).$$

By Theorem 2.6, $I = I(\mathbb{X}) = \langle F_0, F_1, \dots, F_m \rangle$ with

$$\deg F_0 = m, \quad \deg F_1 = d_1 + m - 1 \quad \cdots \quad \deg F_i = d_i + m - i \quad \cdots \quad \deg F_m = d_m$$

and by Theorem 2.10, $J = I(\mathbb{Y}) = \langle G_0, G_1, \dots, G_m, G_{m+1} \rangle$ with

$$\deg G_0 = d_m, \quad \deg G_1 = b - 1 \quad \cdots \quad \deg G_i = b + d_m - d_{m-(i-1)} - i \quad \cdots$$
$$\deg G_m = b + d_m - d_1 - m, \quad \deg G_{m+1} = b$$

(where we note that $m + 1 + l = d_m$).

In fact $F_0 = f_1$, $G_0 = f_2$ (so $F_0 \mid G_0$). Also $F_m = g_1$ and $G_{m+1} = g_2$ so $F_m \mid G_{m+1}$. Thus, $I + J = \langle F_0, F_1, \dots, F_m, G_1, \dots, G_m \rangle$. We now show that this is a minimal generating set for $I + J$. First we show that no F_i can be eliminated from this generating set.

If $F_i \in \langle F_0, \dots, \hat{F}_i, \dots, F_m, G_1, \dots, G_m \rangle$ for some $i = 0, \dots, m$, then

$$F_i = \alpha_0 F_0 + \cdots + \alpha_{i-1} F_{i-1} + \alpha_{i+1} F_{i+1} + \cdots + \alpha_m F_m + \beta_1 G_1 + \cdots + \beta_m G_m$$

for some $\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_m, \beta_1, \dots, \beta_m \in R$. Thus

$$\begin{aligned} M &= \alpha_0 F_0 + \cdots + \alpha_{i-1} F_{i-1} - F_i + \alpha_{i+1} F_{i+1} + \cdots + \alpha_m F_m \\ &= -(\beta_1 G_1 + \cdots + \beta_m G_m) \end{aligned}$$

satisfies, $M \in I \cap J = \langle f_2, g_2 \rangle$.

Hence

$$M = \alpha_0 F_0 + \cdots + \alpha_{i-1}F_{i-1} - F_i + \alpha_{i+1}F_{i+1} + \cdots + \alpha_m F_m = -(\alpha' f_2 + \alpha'' g_2)$$

for some $\alpha', \alpha'' \in R$. In other words,

$$F_i = \alpha_0 F_0 + \cdots + \alpha_{i-1}F_{i-1} + \alpha_{i+1}F_{i+1} + \cdots + \alpha_m F_m + \alpha' f_2 + \alpha'' g_2.$$

Since $(F_0 = f_1) | f_2$ and $(F_m = g_1) | g_2$, $F_i \in \langle F_0, \ldots, \hat{F}_i, \ldots, F_m \rangle$, a contradiction. Hence $F_i \notin \langle F_0, \ldots, \hat{F}_i, \ldots, F_m, G_1, \ldots, G_m \rangle$ for every $i = 0, \ldots, m$.

We now show that no G_j can be eliminated from this set.

Assume $G_j \in \langle F_0, \ldots, F_m, G_1, \ldots, \hat{G}_j, \ldots, G_m \rangle$ for some $j = 1, \ldots, m$. Then

$$G_j = \alpha_0 F_0 + \cdots + \alpha_m F_m + \beta_1 G_1 + \cdots + \beta_{j-1}G_{j-1} + \beta_{j+1}G_{j+1} + \cdots + \beta_m G_m$$

for some $\alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_m \in R$. Thus

$$\begin{aligned} M' &= -(\alpha_0 F_0 + \cdots + \alpha_m F_m) \\ &= \beta_1 G_1 + \cdots + \beta_{j-1}G_{j-1} - G_j + \beta_{j+1}G_{j+1} + \cdots + \beta_m G_m. \end{aligned}$$

Hence $M' \in I \cap J = \langle f_2, g_2 \rangle$ and we can write

$$M' = \beta_1 G_1 + \cdots + \beta_{j-1}G_{j-1} - G_j + \beta_{j+1}G_{j+1} + \cdots + \beta_m G_m = -(\beta' f_2 + \beta'' g_2)$$

for some $\beta', \beta'' \in R$, i.e.

$$G_j = \beta_1 G_1 + \cdots + \beta_{j-1}G_{j-1} + \beta_{j+1}G_{j+1} + \cdots + \beta_m G_m + \beta' f_2 + \beta'' g_2.$$

Since $G_0 = f_2$ and $G_{m+1} = g_2$, $G_j \in \langle G_0, \ldots, \hat{G}_j, \ldots, G_{m+1} \rangle$, a contradiction. Thus $G_j \notin \langle F_0, \ldots, F_m, G_1, \ldots, \hat{G}_j, \ldots, G_m \rangle$ for every $j = 1, \ldots, m$. □

Theorem 2.17. *Let $I + J$ be as in Case* 1 *above. The minimal free resolution of $I + J$, as an R-module, is*

$$0 \to R(-(b + d_m)) \to \bigoplus_{i=1}^{2m+1} R(-b_i) \to \bigoplus_{i=1}^{2m+1} R(-a_i) \to I + J \to 0$$

where

$$\begin{aligned} \bigoplus_{i=1}^{2m+1} R(-a_i) = {} & [R(-m) \oplus R(-(d_1 + m - 1)) \oplus \cdots \oplus R(-(d_i + m - i)) \\ & \oplus \cdots \oplus R(-d_m)] \oplus [R(-b-1)) \oplus \cdots \oplus R(-(b + d_m - d_1 - m))], \end{aligned}$$

$$\begin{aligned} \bigoplus_{i=1}^{2m+1} R(-b_i) = {} & [R(-(b + d_m - m)) \oplus R(-(b + d_m - d_1 - m + 1)) \oplus \cdots \oplus \\ & R(-(b + d_m - d_i - m + i)) \oplus \cdots \oplus R(-b)] \oplus \\ & [R(-(d_m + 1)) \oplus \cdots \oplus R(-(d_1 + m))]. \end{aligned}$$

Proof. Once we have the degrees of a minimal set of generators for $I + J$ the rest follows from [4] since the graded Betti numbers of a minimal resolution are determined by the degrees of the members of a minimal generating set and the socle degree of $R/(I + J)$, and the latter is given by Harima's Theorem 2.15. □

Remark 2.18. For future reference, we will record the numerical data of Theorem 2.17 in a more convenient form.

First, the degrees of the generators of $I + J$ above are (in increasing order):

$$\begin{aligned} m &\le m + (d_1 - 1) \le m + (d_2 - 2) \le \cdots \le m + (d_i - i) \\ &\le \cdots \le m + (d_{m-1} - (m-1)) = d_{m-1} + 1 \le m + (d_m - m) = d_m \\ &\le b - 1 \le b - 1 + (d_m - m) - (d_{m-1} - (m-1)) \le \cdots \\ &\le b - 1 + (d_m - m) - (d_i - i) \le \cdots \le b - 1 + (d_m - m) - (d_1 - 1); \end{aligned}$$

and the degrees of the first syzygies are (in decreasing order):

$$\begin{aligned} b + (d_m - m) &\ge b + (d_m - m) - (d_1 - 1) \ge \cdots \ge b + (d_m - m) - (d_i - i) \\ &\ge \cdots \ge b + (d_m - m) - (d_{m-1} - (m-1)) \ge b \\ &\ge d_m + 1 = (d_m - m) + (m+1) \ge \cdots \ge (d_i - i) + (m+1) \\ &\ge \cdots \ge (d_1 - 1) + (m+1) = d_1 + m. \end{aligned}$$

Notice that if we write the generator degrees as q_i for $i = 1, \ldots, 2m+1$ and the first syzygy degrees as p_i for $i = 1, \ldots, 2m+1$ (in the orders above) then

$p_i = f - q_i$ where $f = d_m + b$ is the degree of the last syzygy.

We now consider the next (and last) case.

Case 2: $a = d_m = m$. In this case $\mathbb{X}$ is a k-configuration of type $(1, 2, \ldots, m)$ embedded in a basic configuration of type (m, b) where $b > m$.

From Theorem 2.6 (for example) we see that all the elements in a minimal generating set for $I = I(\mathbb{X})$ have degree m. In fact, with the notation of Case 1, a minimal generating set for I can be chosen to be $\{F_i \mid i = 0, \ldots, m\}$, where $F_i = L_1, \ldots, L_{m-i} M_{m-(i-1)}, \ldots, M_m$.

The complement, $\mathbb{Y}$, of $\mathbb{X}$ in the basic configuration is also a k-configuration (not weak!) of type $(b - m, b - m + 1, \ldots, b - m + m - 1 = b - 1)$ and thus has generators $G_0, \ldots, G_m$ with degrees $m, b - 1, \ldots, b - 1$. But, one has $G_0 = F_0$ and hence, if $J = I(\mathbb{Y})$ we have $I + J = \langle F_0, \ldots, F_m, G_1, \ldots, G_m \rangle$. With the same proof as in Case 1, one shows that this is a minimal generating set for $I + J$.

After that the steps are completely the same as in Case 1, and, in fact, we can use the statements of both Theorem 2.17 and Remark 2.18 without change for this case also.

Remark 2.19. If $T = (1, h_1, \ldots, h_{j-1}, 1)$ is a vector which satisfies the conditions of Stanley's theorem, and if $(d_1, \ldots, d_m)$ is its associated alignment character then if I is any ideal in $k[x_0, x_1, x_2] = R$ for which $A = R/I$ is a Gorenstein artinian ring with **h**-vector T then it is a well-known fact that $v(I) \leq 2m + 1$ [9, 21]. Thus, Theorem 2.17 tells us that Harima's construction gives an ideal with the maximal possible number of generators, given the **h**-vector.

We shall exploit that fact later.

3. The arithmetic of the h-vector of Gorenstein ideals of codim 3

It would have already been apparent to the reader that our discussions about the **h**-vector of a Gorenstein artinian quotient of $k[x_0, x_1, x_2]$ have largely centered around arithmetic manipulations of the numbers involved.

This was also a key feature in the study of codimension 2 arithmetically Cohen–Macaulay varieties, where arithmetic properties of the integers in the degree matrix of the Hilbert–Burch matrix plays such an important role (see e.g. [6, 12]).

As one might expect in the case of codimension 3, Gorenstein ideals, the skew symmetric matrix of Buchsbaum and Eisenbud [4] plays the same role that the Hilbert–Burch matrix played for the codimension 2 Cohen–Macaulay ideals. In fact, the fundamental "numerical" discussion of this matrix can be found in the paper of Diesel [9]. We will not review all of that work here, but rather summarize those points that we shall need.

So, let I be a homogeneous ideal in $R = k[x_0, x_1, x_2]$ for which $A = R/I = \bigoplus A_i$ is a Gorenstein artinian ring having socle degree j (in short, a Gorenstein ideal in R having socle degree j). Suppose further that $v(I) = n + 1$ and that a minimal generating set for I consists of forms of degree q_i, where $q_0 \leq q_1 \leq \cdots \leq q_n$. Then, I has a minimal free resolution of the form

$$0 \to R(-f) \to \bigoplus_{i=0}^{n} R(-p_i) \to \bigoplus_{i=0}^{n} R(-q_i) \to I \to 0 \qquad (*)$$

where $n + 1$ is odd, $f = j + 3$ and $p_i = f - q_i$.

Consider the infinite sequence $\mathscr{T} = (h_0, h_1, h_2, \ldots, \ldots, \ldots)$ where $h_i = \mathbf{H}(A, i)$ and let $\Delta^3 \mathbf{H}(A, i) = \delta_i$ be the third difference of this sequence. Let T be the vector formed using only the positive integers in $\mathscr{T}$. Then [9, Corollary 2.6] showed that if we set

$$\delta(T) = 2\left\lceil \frac{-\sum_{\delta_i < 0} \delta_i}{2} \right\rceil - 1$$

then $v(I) \geq \delta(T)$, i.e. $\delta(T)$ is a minimum for the number of generators for all Gorenstein ideals having Hilbert function the same as that of A.

Moreover, if we let t be the least integer i for which $h_i < \binom{i+2}{2}$ then $v(I) \leq \mathscr{M}(t) = 2t + 1$ (see e.g. [9, Theorem 3.3] or [21]).

Thus, it is possible to give upper and lower bounds for $\nu(I)$ simply in terms of the **h**-vector T of $A = R/I$.

Also, given T and any odd number ν such that $\delta(T) \le \nu \le \mathcal{M}(T)$ it is possible to find a Gorenstein ideal I in R for which $A = R/I$ has **h**-vector T and $\nu(I) = \nu$ (see [8], [9] or [12]).

Using the notation above, form the sequence of integers $r_i = p_i - q_i$. Then $r_0 \ge r_1 \ge \cdots \ge r_n$. Also, Diesel shows [9, Proposition 3.1]:

(i) the integers r_i are all even or all odd;

(ii) $r_0 > 0$;

(iii) $r_i + r_{n-i+1} > 0$ for $i = 1, \dots, n/2$.

Moreover, whenever $N = \nu(I) = \mathcal{M}(T)$ (the maximum possible) then $r_i + r_{N-i+1} = 2$. Notice that since $p_i = f - q_i$ then $r_i = p_i - q_i = f - 2q_i$. So, $r_i = r_{i'} \Leftrightarrow q_i = q_{i'}$.

In [9, Theorem 3.2] Diesel shows that given any **h**-vector T satisfying Stanley's theorem, then the sequence $\{r_i\}$ obtained from an ideal having the maximal number of generators (i.e. $\mathcal{M}(T)$ generators) is independent of the ideal chosen. In particular, the degrees of the generators of any ideal having **h**-vector T having $\mathcal{M}(T)$ generators, is completely determined by T. Since, in Theorem 2.17 we have seen how to construct such an example for every **h**-vector T satisfying Stanley's theorem, we even know how to calculate those numbers as functions of the alignment character of T and the socle numbers as functions of T i.e. as functions of the alignment character of T and the socle degree.

Furthermore, Diesel also explains how, with this "maximal" sequence of r_i's associated to T, we can deduce the degrees in a minimal generating set for any Gorenstein ideal having Hilbert function T.

The procedure she gives is as follows: given T construct the sequence $r_0 \ge r_1 \ge \cdots \ge r_s$ $(s + 1 = \mathcal{M}(T))$ and suppose that $q_0, q_1, \dots, q_s$ are the degrees of generators associated with this sequence of r_i's. Suppose that $s + 1 > \delta(T)$. Then, there is at least one pair of indices $\{i, i'\}$ for which $r_i = -r_{i'}$. Eliminate two such r_i's from the list and also the degrees of generators associated to them. The resulting set of degrees of generators is actually the set of degrees of a minimal generating set for another Gorenstein ideal with **h**-vector T. We then continue the process for as long as the number of r_i's exceeds $\delta(T)$. In this way we find the "tree" of all possible degree sequences for mimimal generating sets of Gorenstein ideals I for which $A = R/I$ has **h**-vector T.

Example 3.1. Let $T = (1, 3, 6, 10, 12, 12, 10, 6, 3, 1)$. We easily calculate that $\delta(T) = 3$ and $\mathcal{M}(T) = 9$.

From Theorem 2.17 we have a Gorenstein ideal I in R for which $\nu(I) = 9$ and I has generators of degrees: 4, 4, 4, 5, 5, 6, 6, 7, 7. Since $f = j + 3 = 12$ (j the socle degree), the degrees of the syzygies in $(*)$ above are: 8, 8, 8, 7, 7, 6, 6, 5, 5.

So, the sequence $r_0 \ge \cdots \ge r_8$ is:

$$4\ 4\ 4\ 2\ 2\ 0\ 0\ -2\ -2.$$

We see that we can either eliminate a pair $\{2, -2\}$ or a pair $\{0, 0\}$.

We thus have the beginning of our tree:

$$a = \{4, 4, 4, 2, 2, 0, 0, -2, -2\}$$

$$b = \{4, 4, 4, 2, 0, 0, -2\} \qquad c = \{4, 4, 4, 2, 2, -2, -2\}$$

If we are at "b" in the tree, we can either eliminate a pair $\{0, 0\}$ or a pair $\{2, -2\}$, while if we are at "c" our only choice is to eliminate a pair $\{2, -2\}$. Thus we have:

$$a$$

$$b \qquad\qquad c$$

$$d = \{4, 4, 4, 0, 0\} \qquad e = \{4, 4, 4, 2, -2\}$$

At "d" we can only eliminate the pair $\{0, 0\}$, while at "e" we can only eliminate the pair $\{2, -2\}$ obtaining:

$$a$$

$$b \qquad c$$

$$d \qquad e$$

$$z = \{4, 4, 4\}$$

Corresponding to this tree we can now write down all the possible degree sequences for a set of minimal generators of a Gorenstein ideal I with **h**-vector T:

$$a = \{4, 4, 4, 5, 5, 6, 6, 7, 7\}$$

$$b = \{4, 4, 4, 5, 6, 6, 7\} \qquad c = \{4, 4, 4, 5, 5, 7, 7\}$$

$$d = \{4, 4, 4, 6, 6\} \qquad e = \{4, 4, 4, 5, 7\}$$

$$z = \{4, 4, 4\}$$

There is one more "arithmetic" property of codimension 3 Gorenstein ideals that we shall need, i.e. a description of the minimal free resolution of the square of such an ideal. This was discovered by Kustin and Ulrich [16] and reported in [15, Appendix 1]. (Kustin and Ulrich actually construct a resolution for any power of such an ideal.)

Theorem 3.2 (Kustin–Ulrich [16]). *Let $I \subset R = k[x_0, x_1, x_2]$ be a Gorenstein ideal and suppose that I has resolution $(*)$ as above.*

Then I^2 has minimal resolution

$$0 \to \bigoplus_{0 \le i < t \le n} R(-(p_i + p_t)) \to \bigoplus_{\substack{0 \le i, t \le n \\ (i,t) \ne (0,0)}} R(-(p_i + q_t))$$
$$\to \bigoplus_{0 \le i \le t \le n} R(-(q_i + q_t)) \to I^2 \to 0. \qquad (**)$$

Remark. We see from $(**)$ that the graded Betti numbers in the minimal free resolution for I^2 depend only on the degrees in a minimal generating set for I.

Let us now apply these comments, using the information we obtained in Section 2 about generators for Gorenstein ideals.

Remark 3.3. Let $T = (1, h_1, \dots, h_{j-1}, 1)$ be a vector satisfying Stanley's Theorem 2.11. Let $(d_1, \dots, d_m)$ be the alignment character associated to T (see Example 2.12) and so $j \ge 2d_m - 2$.

Since $b = j + 3 - d_m \ge d_m + 1$ we have that $b > d_m$. Thus, Remark 2.18 gives the degrees of the minimal generators (and of first syzygies) of a Gorenstein ideal with **h**-vector T and maximal number of generators.

In this case, then, the integers r_i are:

$$\begin{array}{ll} r_0 = b + d_m - 2m & \\ r_1 = b + d_m + 2(1 - d_1 - m) & r_{m+1} = (d_m + 1) - (b - 1) \\ \vdots & \vdots \\ r_i = b + d_m + 2(i - d_i - m) & r_{2m+1-i} = 2d_i - (b + d_m) + 2(m - i + 1) \\ \vdots & \vdots \\ r_m = b - d_m > 0 & r_{2m} = 2d_1 - (b + d_m) + 2m \end{array}$$

There are a number of simple observations that can be made about these r_i's.

(I) $r_0 \ne -r_i$ for any $i = 1, \dots, 2m$.

Thus, whatever elimination of degrees of generators might be possible, none involves the lowest degree generator (as we would expect!).

(II) If j is even then $r_i \ne 0$ for any i.

To see this notice that $r_i = f - 2q_i = j + 3 - 2q_i$. Thus, if j is even then r_i is odd for every i.

Now suppose $b > d_m + 2$ (i.e. $r_{m+1} < 0$). Then
(III) $r_i = -r_{2m+1-j}$ (for $1 \le i, j \le m$) $\Leftrightarrow (d_i - i) = (d_j - j) + 1$.

Thus, if $b > d_m + 2$ and $(d_i - 2) \neq (d_j - j) + 1$ for any $1 \le i, j \le m$, then no elimination is possible. So, for such a T we must have $\delta(T) = \mathcal{M}(T)$ and the construction of Harima gives the only possible set of generator degrees for a Gorenstein ideal with **h**-vector T.

Example 3.4. Let $T = (1, 3, 6, 7, 8, 8, 8, 7, 6, 3, 1)$. The alignment character associated to T is (1, 2, 5) and so by Theorem 2.15 we have $b = 8 > d_m + 2 = 7$. Thus, any Gorenstein ideal with **h**-vector T has minimal generators having degrees (see Remark 2.18) 3, 3, 3, 5, 7, 9, 9. Thus, if I is a Gorenstein ideal in $R = k[x_0, x_1, x_2]$ with **h**-vector T then I^2 has resolution:

$$\begin{array}{c} 0 \\ \downarrow \\ R^3(-6) \oplus R^3(-8) \oplus R^3(-10) \oplus R^7(-12) \oplus R^2(-14) \oplus R^2(-16) \oplus R(-18) \\ \downarrow \\ R^6(-7) \oplus R^5(-9) \oplus R^6(-11) \oplus R^{14}(-13) \oplus R^6(-15) \oplus R^5(-17) \oplus R^6(-19) \\ \downarrow \\ R^6(-6) \oplus R^3(-8) \oplus R^4(-10) \oplus R^7(-12) \oplus R^3(-14) \oplus R^2(-16) \oplus R^3(-18) \\ \downarrow \\ I^2 \\ \downarrow \\ 0 \end{array}$$

4. The dimension of the tangent space to $\mathcal{G}or(T)$ in codimension 3

To fix the notation in this section, let $T = (1, h_1, \ldots, h_{j-1}, 1)$ be a symmetric sequence of length j with $h_1 \le 3$ and satisfying the hypothesis of Stanley's theorem (Theorem 2.11). So $\mathbf{Gor}(T) \neq \emptyset$. In fact, Diesel [9] has shown that $\mathbf{Gor}(T)$ is even irreducible in this case.

Let $[T]$ be the collection of all the Gorenstein ideals in $R = k[x_0, x_1, x_2]$ for which T is the **h**-vector of R/I. Let $S = k[y_0, y_1, y_2]$ and let $F_I \in \mathbb{P}(S_j)$ be the point corresponding to I in the Macaulay correspondence mentioned in Section 1. Then $\mathbf{Gor}(T)$ is the collection of F_I corresponding to $I \in [T]$.

Conjecture 4.1. With the notation above and also that of Theorem 1.4 we have $\dim_k \mathcal{T}_{F_I}$ is independent of $I \in [T]$. Equivalently (in view of Theorem 1.4) if $I, J \in [T]$ then $\dim_k I_j^2 = \dim_k J_j^2$.

Remark 4.2. Since we always have

$$\text{dimension of } \mathbf{Gor}(T) \le \text{dimension of } \mathcal{G}or(T) \le \dim_k \mathcal{T}_{F_I}$$

for any $F_I \in \mathbf{Gor}(T)$, it follows that IF Conjecture 4.1 is true for T and if

$$\text{dimension of } \mathbf{Gor}(T) = \dim_k \mathscr{T}_{F_I} \text{ for any } F_I \in \mathbf{Gor}(T) \qquad (*)$$

then there is equality for every $F_I \in \mathbf{Gor}(T)$ and we obtain that $\mathbf{Gor}(T) = \mathscr{G}or(T)$ and that both are smooth.

We prove Conjecture 4.1 for all the classes of T considered in [15, Theorem 3.1A and 3.4A] and thereby extend the results of those theorems to all of the points of $\mathbf{Gor}(T)$. At the end of this section we give some further conjectures (one of which would imply Conjecture 4.1) and gather some evidence for these conjectures as well.

We now describe those T we shall consider in this section and for which we shall prove Conjecture 4.1.

Definition 4.3. We define $H(s,j) := (1, h_1, \dots, h_{j-1}, 1)$, where $h_i = \min\{s, \dim_k R_i, \dim_k R_{j_i}\}$ for $0 \le i \le j$.

Consider $\lfloor j/2 \rfloor = \tau$. There are then two possibilities for s:
(a) $s \ge \binom{r+2}{2}$. Then,

$$\text{if } j \text{ is even } H(s,j) := \left(1, 3, \dots, \binom{\tau+1}{2}, \binom{\tau+2}{2}, \binom{\tau+1}{2}, \dots, 3, 1\right)$$

$$\text{if } j \text{ is odd } H(s,j) := \left(1, 3, \dots, \binom{\tau+1}{2}, \binom{\tau+2}{2}, \binom{\tau+2}{2}, \binom{\tau+1}{2}, \dots, 3, 1\right).$$

(b) $s < \binom{\tau+2}{2}$. In this case, let τ' be the least integer for which

$$\binom{\tau'+2}{2} \le s < \binom{\tau'+3}{2}.$$

Then

$$H(s,j) := \left(1, 3, \dots, \binom{\tau'+2}{2}, s, \dots, s, \binom{\tau'+2}{2}, \dots, 3, 1\right).$$

Theorem 4.4. *Conjecture* 4.1 *is true for* $T = H(s,j)$ *for any* s *and* j.

Proof. We may as well write

$$T = \left(1, 3, \dots, \binom{t+2}{2}, s, \dots, s, \binom{t+2}{2}, \dots, 3, 1\right)$$

where $s = \binom{t+3}{2} - a$, $0 < a \le t+2$.

For such a T, the alignment character is:

when $a = 1$	$(2, 3, \dots, t+1, t+2)$
when $2 \le a \le t \le 1$	$(1, 2, \dots, a-1, a+1, \dots, t+1, t+2)$
and when $a = t+2$	$(1, 2, \dots, t, t+1)$.

Clearly we must have $j \geq 2t$, but

(α) if $j = 2t$ then $s = \binom{t+2}{2}$ and

$$H(s,j) = 1 \;\; 3 \;\; \cdots \;\; \binom{t+1}{2} \; \binom{t+2}{2} \; \binom{t+1}{2} \;\; \cdots \;\; 3 \;\; 1;$$

while

(β) if $j = 2t + 1$ then we must also have $s = \binom{t+2}{2}$, but now

$$H(s,j) = 1 \;\; 3 \;\; \cdots \;\; \binom{t+1}{2} \; \binom{t+2}{2} \; \binom{t+2}{2} \; \binom{t+1}{2} \;\; \cdots \;\; 3 \;\; 1.$$

In both cases, $a = t + 2$. We first deal with these two special cases.

Case α: In this case it is easy to see that if $I \in [T]$ then there is only one possible set of degrees in a minimal generating set for I, namely $2t + 3$ generators of degree $t + 1$. Thus, (in the notation of Theorem 2.17 and Remark 2.18) we get that all of the first syzygies of I (there must be $2t + 3$ of them) have degree $t + 2$.

Hence, for any $I \in [T]$ the minimal free resolution of I^2 must be:

$$0 \to R^{(2t+2)(2t+3)/2}(-(2t+4)) \to R^{(2t+3)^2-1}(-(2t+3))$$
$$\to R^{(2t+3)(2t+4)/2}(-(2t+2)) \to I^2 \to 0.$$

Since $j = 2t$ we obtain

$$\dim_k (I^2)_{2t} = 0 \text{ and so } \dim_k (R/I^2)_{2t} = \dim_k R_{2t} = \binom{2t+2}{2}.$$

It follows that

$$\dim_k \mathscr{T}_{F_I} = \binom{2t+2}{2} - 1$$

in this case.

Case β: Now we have $j = 2t + 1$ and $s = \binom{t+2}{2}$ and (in the notation of Theorem 2.17) $b = 2t + 3$. Any $I \in [T]$ with the maximal number of generators has $t + 2 = a$ generators of degree $t + 1$ and $t + 1$ generators of degree $t + 2$.

We write down the data of degrees and syzygies in the following tableau (see Section 3 for the notation):

$$\overbrace{t+1 \cdots\cdots t+1}^{a\text{-times}} \qquad \overbrace{t+2 \cdots\cdots t+2}^{(t+1)\text{-times}} \qquad (q_i)$$

$$\underbrace{t+3 \cdots\cdots t+3}_{a\text{-times}} \qquad \underbrace{t+2 \cdots\cdots t+2}_{(t+1)\text{-times}} \qquad (p_i)$$

$$\underbrace{2 \cdots\cdots 2}_{a\text{-times}} \qquad \underbrace{0 \cdots\cdots 0}_{(t+1)\text{-times}} \qquad (r_i)$$

Thus, any ideal $I \in [T]$ will have

$$\begin{array}{ll} a = t + 2 & \text{generators of degree } t + 1; \\ (t + 1) - 2v (v \geq 0) & \text{generators of degree } t + 2. \end{array}$$

It follows that for any ideal $I \in [T]$, the resolution of I^2 is:

$$0 \to F_2 \to F_1 \to F_0 \to I^2 \to 0,$$

where

$$\begin{aligned} F_0 &= R^{a(a+1)/2}(-(2t+2)) \oplus R^{a(t+1-2v)}(-(2t+3)) \\ &\quad \oplus R^{(t+1-2v)(t+2-2v)/2}(-(2t+4)); \\ F_1 &= R^{a(t+1-2v)}(-(2t+3)) \oplus R^{(t+1-2v)^2}(-(2t+4)) \oplus R^{a(t+1-2v)-1}(-(2t+5)); \\ F_2 &= R^{(t-2v)(t+1-2v)/2}(-2t+4)) \oplus R^{a(t+1-2v)}(-(2t+5)) \\ &\quad \oplus R^{a(a-1)/2}(-(2t+6)). \end{aligned}$$

Thus $\dim_k(I^2)_{2t+1} = 0$ and so in this case we have,

$$\dim_k \mathscr{F}_{F_t} = \dim_k R_{2t+1} - 1 = \binom{2t+3}{2} - 1.$$

Aside: If we consider $T = (1, 3, 6, 6, 3, 1)$ (which fits into the case just considered for $t = 2$) we find that $\delta(T) = 5$ and $\mathscr{M}(T) = 7$. So, there are two possible degree sequences for minimal generating sets of ideals $I \in [T]$, i.e. v can be either 0 or 1 for this T. Thus the resolutions of I^2 for $I \in [T]$ can be different. But, a quick look at the resolutions above show that $\mathbf{H}(R/I^2, -)$ is independent of $I \in [T]$, i.e. independent of v. We shall return to this comment later.

We now move to the general case,

$$j \geq 2t + 2, \quad T = H(s, j), \quad s = \binom{t+3}{2} - a, \quad 0 < a \leq t + 2.$$

We may apply the procedure of Theorem 2.17 (see also Remark 2.18) to produce an ideal $I \in [T]$ with the maximum possible number of generators. For such an ideal we have the following data (see also Section 3 for notation):

$$\begin{array}{ll} \overbrace{t+1 \cdots\cdots t+1}^{a\text{-times}} & \overbrace{t+2 \cdots\cdots t+2}^{(t-a+2)\text{-times}} \\ & \\ \underbrace{t+n \cdots\cdots t+n}_{(t-a+2)\text{-times}} & \underbrace{t+n+1 \cdots\cdots t+n+1}_{(a-1)\text{-times}} \end{array} \qquad (q_i)$$

$$\overbrace{t+n+2\cdots\cdots t+n+2}^{a\text{-times}} \quad \overbrace{t+n+1\cdots\cdots t+n+1}^{(t-a+2)\text{-times}}$$

$$\underbrace{t+3\cdots\cdots t+3}_{(t-a+2)\text{-times}} \quad \underbrace{t+2\cdots\cdots t+2}_{(a-1)\text{-times}} \qquad (p_i)$$

$$\overbrace{n+1\cdots\cdots n+1}^{a\text{-times}} \quad \overbrace{n-1\cdots\cdots n-1}^{(t-a+2)\text{-times}}$$

$$\underbrace{3-n\cdots\cdots 3-n}_{(t-a+2)\text{-times}} \quad \underbrace{1-n\cdots\cdots 1-n}_{(a-1)\text{-times}} \qquad (r_i)$$

We observed that the generator degree sequence (q_i') and syzygy degree sequence (p_i') of any ideal $I' \in [T]$ is obtained from that above by eliminating pairs of generator degrees (and syzygy degrees) corresponding to pairs of $\{\alpha, -\alpha\}$ among the r_i's.

Thus, any other ideal $I \in [T]$ will have:

a generators of degree $t+1$;

$(t-a+2)-v$ generators of degree $t+2$;

$(a-1)-v$ generators of degree $t+n+1$;

and, if $n \neq 3$; $t-a+2$ generators of degree $t+n$;

and if $n=3$; $(t-a-2)-2\eta$ generators of degree $t+n$;

where v and η are ≥ 0.

As in the special cases above, the idea is to use Theorem 3.2 to calculate $\dim_k I_j^2$ for $I \in [T]$ and to show that this dimension does not depend on v and η and only depends on t, a and n, i.e. only on T.

We will carry through the details in the case $j = 2t+n$, $n > 3$ and only report on the conclusions for the cases $n = 2, 3$. The ideas are identical in those cases as well.

So, let $j = 2t+n$, $n>3$. Then, from Theorem 3.2 we have, for $I \in [T]$,

$$0 \to (F_2)_j \to (F_1)_j \to (F_0)_j \to (I^2)_j \to 0,$$

where

$$(F_0)_j = R^{a(a+1)/2}(-(2t+2))_j \oplus R^{a(t-a+2-v)}(-(2t+3))_j$$
$$\oplus R^{(t-a+2-v)(t-a+3-v)/2}(-(2t+4))_j;$$

Note. F_0 may have other free summands $R(-m)$, but the others all have $m>j$ and so $R(-m)_j = R_{j-m} = (0)$. The same comment applies to F_1 and F_2.

$$(F_1)_j = R^{a(a-1-v)}(-(2t+3))_j \oplus R^{a(t-a+2)+(a-1-v)(t-a+2-v)}(-(2t+4))_j$$
$$\oplus R^{(t-a+2)^2-(t-a+2)v}(-(2t+5))_j$$

and

$$(F_2)_j = R^{(a-1-v)(a-2-v)/2}(-(2t+4))_j \oplus R^{(t-a+2)(a-1-v)}(-(2t+5))_j$$
$$\oplus R^{(t-a+2)(t-a+1)/2}(-(2t+6))_j.$$

We can see from these formulae that we must consider, separately, the cases when $n = 4, 5$, or $n \geq 6$. We do that now.

Case $n = 4$: So, we have $j = 2t + 4$. Then

$$\begin{aligned}\dim_k(I^2)_j = & \frac{a(a+1)}{2}\dim_k R_2 + a(t-a+2-v)\dim_k R_1 \\ & + \frac{(t-a+2-v)(t-a+3-v)}{2} - a(a-1-v)\dim_k R_1 \\ & - a(t-a+2) - (a-1-v)(t-a+2-v) \\ & + \frac{(a-1-v)(a-2-v)}{2}.\end{aligned}$$

A simple calculation shows that this last is equal to

$$6 + \tfrac{1}{2}t^2 + \tfrac{7}{2}t + 3a$$

which is independent of v, as we wanted to show.

It follows that

$$\dim_k \mathscr{T}_{F_t} + 1 = \mathbf{H}(R/I^2, 2t+4) = \binom{2t+6}{2} - \left(6 + \frac{1}{2}t^2 + \frac{7}{2}t + 3a\right)$$

and so

$$\dim_k \mathscr{T}_{F_t} = \tfrac{3}{2}t^2 + \tfrac{15}{2}t + 8 - 3a.$$

Case $n = 5$: We set $j = 2t + 5$, then

$$\begin{aligned}\dim_k(I^2)_j = & \frac{a(a+1)}{2}\dim_k R_3 + a(t-a+2-v)\dim_k R_2 \\ & + \frac{(t-a+2-v)(t-a+3-v)}{2}\dim_k R_1 - a(a-1-v)\dim_k R_2 \\ & - (\dim_k R_1)[a(t-a+2) + (a-1-v)(t-a+2-v)] \\ & + \frac{(a-1-v)(a-2-v)}{2}\dim_k R_1 \\ & - (t-a+2)^2 + (t-a+2)v + (t-a+2)(a-1-v).\end{aligned}$$

A simple calculation shows that this last is equal to

$$12 + \tfrac{1}{2}t^2 + \tfrac{11}{2}t + 3a$$

which is independent of v, as we wanted to show.

It follows that

$$\dim_k \mathcal{T}_{F_l} + 1 = \mathbf{H}(R/I^2, 2t+5) = \binom{2t+7}{2} - \left(12 + \frac{1}{2}t^2 + \frac{11}{2}t + 3a\right)$$

and so

$$\dim_k \mathcal{T}_{F_l} = \frac{3}{2}t^2 + \frac{15}{2}t + 8 - 3a.$$

Note that this is the same result we obtained for $j = 2t + 4$.

Case $n \geq 6$: Now $j = 2t + n$ where $n \geq 6$. Then precisely the same sort of computation as above gives that

$$\dim_k \mathcal{T}_{F_l} = \tfrac{3}{2}t^2 + \tfrac{15}{2}t + 8 - 3a$$

which is independent of both v and n.

Aside: So, we see that the calculation of $\dim_k \mathcal{T}_{F_l}$ depends only on t and a when $j = 2t + n$ and $n \geq 4$. This is not same value we will find when $n = 2$ and $n = 3$.

Case $n = 3$: Since $j = 2t + 3$, the free resolution of $(I^2)_j$ is now given by

$$(F_0)_j = R^{a(a+1)/2}(-(2t+3))_j \oplus R^{a(t-a+2-v)}(-2t+3))_j$$

while

$$(F_1)_j = R^{a(a-1-v)}(-(2t+3))_j \text{ and } (F_2)_j = (0).$$

(Notice that η does not even appear in this part of the resolution for I^2.)

So,

$$\begin{aligned}\dim_k (I^2)_j &= (\dim_k R_1)\left(\frac{a(a+1)}{2}\right) + a(t-a+2-v) - a(a-1-v)\\ &= \tfrac{1}{2}a^2 + \tfrac{9}{2}a + at.\end{aligned}$$

Thus

$$\dim_k \mathcal{T}_{F_l} + 1 = \mathbf{H}(R/I^2, 2t+3) = \binom{2t+5}{2} - \left(\frac{1}{2}a^2 + \frac{9}{2}a + at\right)$$

and so

$$\dim_k \mathcal{T}_{F_l} = 2t^2 + (9-a)t + (9 + \tfrac{1}{2}a^2 - \tfrac{9}{2}a).$$

Case $n = 2$: Now we have $j = 2t + 2$. Omitting the details, we get

$$\dim_k \mathcal{T}_{F_l} = 2t^2 + 7t + (5 - \tfrac{1}{2}a^2 - \tfrac{1}{2}a).$$

This completes the proof of the theorem. □

Corollary 4.5. *Let* $T = H(s,j)$ *for* $s = \binom{t+3}{2} - a$ *where* $0 < a \leq t + 2$. *If* $j = 2t + n$ *where* $n \geq 4$, *then* $\overline{\mathcal{G}or(T)} = Sec_{s-1}(v_j(\mathbb{P}^2))$.

Proof. We first observe that the dimension of the variety $Sec_{s-1}(v_j(\mathbb{P}^2)) = 3s - 1$. This is so because the only deficient secant varieties of Veronese Surfaces occur either when $j = 2$ or when $j = 4$ and $s - 1 = 4$ ([1], see [10] for a discussion) and these are cases not covered by the hypothesis.

One then checks that

$$3s - 1 = \tfrac{3}{2}t^2 + \tfrac{15}{2}t + 8 - 3a.$$

This is precisely the dimension we found for $\mathcal{T}_{F_I}$ for every $I \in [T]$ and that is enough to finish the proof. □

Remark 4.6. (a) This corollary should be compared with Theorem 3.1A and Theorem 3.4A of [15].

(b) It would be interesting to know if, for the T of the corollary, $\mathcal{G}or(\leq T) = \overline{\mathcal{G}or(T)}$ (see [15, Example 5.8]).

(c) The following example is interesting and points out some things that we still do not know. Let $s = 7$, i.e. $s = 10 - 3$ so that $t = 2$ and $a = 3$. Choose $j = 9 = 2 \cdot 2 + 5$ so that Corollary 4.5 applies to $T = (1, 3, 6, 7, 7, 7, 7, 6, 3, 1)$.

There are two interesting catalecticant matrices associated to this example; the first, A (a 10×28 matrix), describes the third partials of a general form of degree 9 and the second, B (a 15×21 matrix), describes the fourth partials of a general form of degree 9.

We know that $\mathcal{G}or(\leq T)$ is described by the ideal $I = I_8(A) + I_8(B)$ and we have shown that the primary decomposition of $I = \wp \cap q$, where $\wp$ is the prime ideal which describes the variety $Sec_6(v_9(\mathbb{P}^2)) \subseteq \mathbb{P}(S_9)$ $(S = k[y_0, y_1, y_2])$ and q is primary for the irrelevant ideal in the homogeneous coordinate ring of $\mathbb{P}(S_9)$.

It would be interesting to know if I is a saturated ideal (we believe that it is) and show that q does not appear in the primary decomposition of I.

It would also be interesting to know if $I_8(A) \subseteq I_8(B)$ (a reasonable assumption since $rkB \leq 7 \Rightarrow rkA \leq 7$). But, this last problem is about equations and not just about the underlying reduced variety and so the statement about ranks remains only an "indication".

This is just one of a family of such questions suggested by Corollary 4.5.

In view of the corollary, the only $T = H(s, j)$ for which we have not yet proved that $\mathcal{G}or(T)$ is smooth and equal to $\mathbf{Gor}(T)$ are those T for which $s = \binom{t+3}{2} - a$, where $0 < a \leq t + 2$ and $j = 2t + n$, where $0 \leq n \leq 3$.

Remark 4.7. We have seen that if $j = 2t$ or $2t + 1$ then the only s we need consider is $s = \binom{t+2}{2}$, corresponding to $a = t + 2$. But, in both of those cases it is easy to see that $\mathbf{Gor}(\leq T) = \mathcal{G}or(\leq T) = \mathbb{P}(S_j)$ which is of dimension

$$\binom{j+2}{2} = \binom{2t+2}{2} \quad (\text{if } j = 2t) \quad \text{or} \quad \binom{2t+3}{2} \quad (\text{if } j = 2t + 1).$$

So, the only cases still left to consider arise when either $j = 2t + 2$ or $j = 2t + 3$.

Example 4.8. Consider $T = (1, 3, 5, 3, 1)$ (cf. Example 1.2). In this case $s = 6 - 1 = 5$ and so $t = 1$, $a = 1$ and $j = 4 = 2t + 2$.

From Theorem 4.4 we obtain that for $I \in [T]$, $\dim_k \mathcal{T}_{F_t} = 13$.

We can conclude, then, that $\overline{\mathcal{G}or(T)} = Sec_4(v_4(\mathbb{P}^2))$. Note that this is one of the deficient secant varieties to a Veronese Surface. In fact, a simple check (using Theorem 4.4) shows that all the inequalities of Example 1.3(3) are equalities and we obtain that $\mathbf{Gor}(T) = \mathcal{G}or(T)$ for all the T of that example.

We can say something more in the case $j = 2t + 3$.

Proposition 4.9. *Let* $s = \binom{t+3}{2} - a$, *where* $0 < a \leq t + 2$ *and let* $j = 2t + 3$.

If $a = 1, 2, t + 1$ *or* $t + 2$ *then* $\mathbf{Gor}(T) = \mathcal{G}or(T)$ *and so* $\mathbf{Gor}(T)$ *is smooth in these cases.*

Proof. In these cases Iarrobino and Kanev have found the dimension of $\mathbf{Gor}(T)$ [15, Theorem 3.4B] and it is the same as the dimension of the tangent space to $\mathcal{G}or(T)$ at every one of its closed points (from Theorem 4.4). In view of Remark 4.2 that is enough to prove the proposition. □

Remark 4.10. It would be very interesting to know if $\mathbf{Gor}(T) = \mathcal{G}or(T)$ for every $T = H(s, j)$.

If $T = H(s, j)$ it is not hard to show (although the calculations are tedious) that whenever I, $J \in [T]$, then I^2 and J^2 have the same Hilbert function. It is reasonable, therefore, to conjecture:

Conjecture 4.11. If I and J are homogeneous ideals in $R = k[x_0, \ldots, x_3]$ of codimension 3 for which R/I and R/J are Gorenstein with the same Hilbert function then R/I^2 and R/J^2 have the same Hilbert function.

Having made that conjecture, it makes sense to conjecture the analogous result for codimension two Cohen Macaulay ideals. More precisely,

Conjecture 4.12. Let I and J be ideals of points in $\mathbb{P}^2$. If R/I and R/J have the same Hilbert function then the same is true for R/I^2 and R/J^2.

Remark 4.13. (a) A weaker conjecture then Conjecture 4.12 in which we only assume that R/I and R/J are arithmetically Cohen–Macaulay and of codimension 2 is false, as an example of Peterson shows (see [19]).

(b) A proposition of Iarrobino–Kanev [15, Theorem 3.6.1] gives some evidence for the truth of Conjecture 4.12. They prove the weaker result that Conjecture 4.12 is true if I and J are both the ideals of a set of points in $\mathbb{P}^2$ with generic Hilbert function *and* minimum number of generators possible for tht Hilbert function.

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